Self-similar bootstrap of divergent series

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A method is developed for calculating effective sums of divergent series. This approach is a variant of the self-similar approximation theory. The interesting component here is using an algebraic transformation with a power providing maximal stability of the self-similar renormalization procedure. The latter is to be repeated as many times as necessary in order to convert into closed self-similar expressions all sums from the series considered. This multiple and complete renormalization is called a self-similar bootstrap. The method is illustrated by several examples from statistical physics. [S1063-651X(97)14805-X]

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I. INTRODUCTION

The most powerful analytic tool for solving realistic, and therefore difficult, problems in theoretical physics and applied mathematics is perturbation theory. However, perturbation series are notoriously badly behaved: in the majority of interesting cases they are divergent. A variety of mathematical techniques have been invented to assign a finite value to the sum of a divergent series. Such techniques are generally referred to as renormalization or summation methods. These methods are not only useful to theoretical and mathematical physicists, but are crucial because they provide a way to recover physical information from perturbative calculations. Probably the most common technique used to assign a meaningful value to a divergent series is the Padé summation [1]. Using the latter, one converts a formal power series to a continued fraction. Truncating this fraction at successive orders, one obtains rational functions called Padé approximants. To reach a reasonable accuracy of such approximants, one usually needs to have tens of terms in a perturbative series.

Recently, a method was suggested [2–4] that permits us to ascribe meaningful values to the limits of divergent sequences by exploiting just a few terms of perturbative series. This approach, called the self-similar approximation theory [2–4], is based on the following ideas.

First, one has to incorporate into the considered sequence additional functions whose role is to renormalize the sequence, making it convergent. These functions, because of their role, are called governing or control functions. The latter are to be defined from fixed-point conditions [2–4]. There are several ways of introducing such control functions. One natural way is to include them in the initial approximation [5,6]. Fixed-point equations may be written in the form of the minimal-difference condition [5–7], or the minimal-sensitivity condition [8–14]. A condition close to the latter type has also been used in the potential envelope method [15,16]. In two simple cases, of zero-dimensional and one-dimensional anharmonic oscillators, the control functions were found analytically for an arbitrary perturbation order [17–20] by requiring the convergence of renormalized perturbation theory. Another way of introducing control functions is by using a scaled basis [21–24], in which scaling parameters, playing the role of such functions, are given by analytical expressions with coefficients adjusted empirically from the convergence of a numerical iterative procedure. One can also define scaling parameters without using their analytical representation, directly from a numerical search providing, the convergence of an iterative method [25–29]. In the present paper we suggest a way of introducing control functions that, to our knowledge, different from all variants mentioned.

Another step in the self-similar approximation theory is to construct an approximation cascade whose trajectory is bijective to the approximation sequence considered [30–32]. In this paper we construct such a cascade not for the sequence itself but for its transform. Of course, the idea of considering a transformed series instead of an initial one is not new. This is, e.g., the basis of the known Borel summation. Another example is the use of Chebyshev transforms instead of straightforward power-series representation [33]. What we believe is new in our approach is the use of a power-law algebraic transform with powers playing the role of control functions.

The constructed approximation cascade is embedded into an approximation flow. Integrating the evolution equation of the flow, we obtain a self-similar approximation [2–4,30–32].

In this paper we define control functions from the principle of maximal stability of the approximation cascade trajectories. This is done by minimizing the absolute values of mapping multipliers, which is equivalent to a quasifixed-point condition [30–32], since multipliers tend to zero when approaching a stable fixed point.

The plan of the paper is as follows. In Sec. II algebraic transforms are introduced, and the main steps of the self-similar approximation theory are sketched, not going into mathematical details which can be found in Refs. [2–4,30–32,34]. In Sec. III the procedure of the self-similar bootstrap is defined, consisting of multiple self-similar renormalizations of all sums entering into a given series. In Sec. IV a particular case of the self-similar bootstrap is considered, leading to a nice representation in the form of multiple exponentials. The following sections illustrate the approach by various examples, emphasizing the generality of the method that can be applied to problems of quite a different nature.

II. ALGEBRAIC TRANSFORMS

Assume that we are interested in finding a function $f(x)$ of a real variable $x \in (-\infty, +\infty)$. Without loss of generality, the function $f(x)$ may be considered to be real, since the case of a complex function can be always reduced to that of two real functions. Let perturbation theory give for the function $f(x)$ approximations $p_k(x)$, with $k=0,1,2,\ldots$. enumerating the approximation number. The standard form of $p_k(x)$ is a series in powers, not necessarily integers, of x. The series can even include logarithms, since the latter can always be presented, using the replica trick, as an expression containing a noninteger power.

The algebraic transform is defined as

$$P_k(x,s) = x^s p_k(x), \tag{1}$$

with s real positive. This transform changes the powers of the series $p_k(x)$, thus changing the convergence properties of the latter. Effectively, the approximation order increases from k to $k+s$ as a result of Eq. (1). The transform inverse to Eq. (1) is

$$p_k(x) = x^{-s} P_k(x,s). \tag{2}$$

To construct an approximation cascade, we proceed as follows. Define the expansion function $x=x(f,s)$ by the equation

$$P_0(x,s) = f, \tag{3}$$

where P_0 is the first available expression from Eq. (1). Substituting $x(f,s)$ back into Eq. (1), we obtain

$$y_k(f,s) \equiv P_k(x(f,s),s). \tag{4}$$

The left-hand side of Eq. (4) represents a point of the approximation-cascade trajectory corresponding to approximation (1). The transformation inverse to Eq. (4) reads

$$P_k(x,s) = y_k(P_0(x,s),s). \tag{5}$$

Function (4) realizes the endomorphism

$$y_k(f,s)\colon \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}.$$

Consider the family $\{y_k\colon k \in \mathbb{Z}_+\}$ as a dynamical system in discrete time. Since the trajectory of this dynamical system, according to Eqs. (4) and (5), is bijective to the approximation sequence $\{P_k\}$, this system was called [30–32] the approximation cascade. In order to simplify the consideration, let us pass from discrete time to continuous time. To this end, embed the approximation cascade into an approximation flow,

$$\{y_k\colon k \in \mathbb{Z}_+\} \subset \{y(t,\ldots)\colon t \in \mathbb{R}_+\},$$

which means that the trajectory $\{y(t,f,s)\}$ of the flow has to pass through all points $\{y_k(f,s)\}$ of the cascade trajectory.

The evolution equation

$$\frac{\partial}{\partial t} y(t,f,s) = v(y(t,f,s),s) \tag{6}$$

for the approximation flow, where $v(f,s)$ is the velocity field, can be integrated for an arbitrary time interval, say, from $t=k-1$ to $t=k^*$, which gives

$$\int_{y_{k-1}}^{y_k^*} \frac{df}{v(f,s)} = k^* - k + 1; \tag{7}$$

here

$$y_k = y(k,f,s), \quad y_k^* = y(k^*,f,s).$$

The upper limit in Eq. (7) corresponds, according to Eq. (5), to an approximation

$$P_k^*(x,s) = y(k^*, P_0(x,s), s). \tag{8}$$

The moment $t=k^*$ is chosen so that to reach approximation (8) by the minimal number of steps. That is, we require that the right-hand side of Eq. (7) be minimal,

$$t_k^* \equiv \min(k^* - k + 1). \tag{9}$$

Under condition (9), expression (8) is called the self-similar approximation.

To find Eq. (8) explicitly, we need to concretize in Eq. (7) the velocity field $v(f,s)$. This can be done by the Euler discretization of Eq. (6), yielding the finite difference

$$v_k(f,s) = y_k(f,s) - y_{k-1}(f,s). \tag{10}$$

Thus, using Eq. (5), the evolution integral (7) can be written as

$$\int_{P_{k-1}}^{P_k^*} \frac{df}{v_k(f,s)} = t_k^*, \tag{11}$$

where

$$P_k = P_k(x,s), \quad P_k^* = P_k^*(x,s).$$

When no additional restrictions are imposed, the minimal number of steps for reaching a quasifixed point is, evidently,

$$\operatorname{abs}\min t_k^* = 1. \tag{12}$$

Additional restrictions can be of different types. For example, if the value of the sought function at some point x_0 is known, we may require that the found approximation would coincide at this point with the given exact value. Looking for an approximation in the class of functions with a prescribed symmetry is another way of imposing restrictions. In some cases the asymptotic behavior of the sought function at $x \to 0$ and $x \to \infty$ may be available. Then, we require that the correct asymptotic properties also play the roles of such additional constraints. In what follows, we shall concretize these variants by illustrating them with explicit examples.

In this way, the sole quantity that is not yet defined is the parameter s of transformation (1). Recall that our aim is to find an approximate fixed point of the cascade trajectory, a quasifixed point, which, by construction, represents the sought function. Therefore, the power s of the transform in Eq. (1) is to be chosen so as to force the trajectory of the approximation dynamical system to approach an attracting

fixed point. Recall that s here is nothing but a kind of control function, so that it is to be defined by a fixed-point condition. As is discussed in Sec. I, there are several forms of equations defining fixed points. Here we opt for a condition following from the analysis of fixed-point stability properties.

Considering the mapping given by the approximation cascade, we may introduce the mapping multipliers

$$\mu_k(f,s)=\frac{\partial}{\partial f}y_k(f,s). \tag{13}$$

This is related to the local Lyapunov exponent λ_k through the formula $\lambda_k=(1/k)\ln|\mu_k|$. Consequently, $\mu_k\sim e^{\lambda_k k}$. If at increasing time, here at $k\to\infty$, the trajectory approaches an attracting fixed point, then $\lambda_k\to\lambda<0$. This implies that the multiplier $\mu_k\to 0$, as $k\to\infty$. Another quantity related to multiplier (13) is the predictability time [35], which can be defined as $\tau_k\approx|\lambda_k|^{-1}$, or $\tau_k\approx|k/\ln|\mu_k||$. This is the characteristic time during which the motion along the cascade trajectory effectively approaches a fixed point. When the latter is attractive—that is, when the limit of the local Lyapunov exponent λ_k, as $k\to\infty$, tends to a negative value $\lambda<0$—then μ_k tends to zero, and at the same time, larger absolute values $|\lambda_k|$ lead to smaller characteristic times τ_k.

These properties show that, the closer we are to a fixed point, the smaller is the absolute value of the multiplier (13). Hence we may define the control function s as that providing the minimum of the multiplier. Instead of the multiplier (13), as a function of the variable f, it may be more convenient to pass to its image

$$m_k(x,s)=\mu_k(F_0(x,s),s), \tag{14}$$

which is a function of the variable x. Then the control function $s=s_k(x)$ is defined by the equation

$$|m_k(x,s_k(x))|=\min_s|m_k(x,s)|. \tag{15}$$

Because the minimization of the multiplier makes the trajectory more stable, we can call Eq. (15) *the principle of maximal stability*. And the so defined control function $s_k(x)$ can be termed the *stabilizing control function*, or, for brevity, the *stabilizer*. Note, for comparison, that another definition of the fixed point would be to require velocity (10) to be zero, which is exactly the minimal-difference condition [5,6]. After the stabilizer $s_k(x)$ is found from Eq. (15), we substitute it into Eq. (8), and, using the inverse transformation (2), we obtain the *self-similar approximation*

$$f_k^*(x)=x^{-s_k(x)}P_k^*(x,s_k(x)) \tag{16}$$

for the sought function.

At the end of this section, let us note that the choice of control functions from fixed-point equations is rigorously justified when the fixed point is stable, that is, if $|\mu_k|<1$. When at some point k the trajectory becomes unstable, we have to stop the calculational procedure at the last stable point. Another possibility is to restructure the considered perturbation series. For instance, the instability of the procedure often happens when we are trying to construct self-similar approximation for a divergent function. Assume that we are dealing with such a function $f(x)$ which diverges as $x\to\infty$. Then the self-similar renormalization procedure may become unstable at large x. To avoid the instability, we can either consider the function $f^{-1}(x)$, or can rewrite the series in powers of $1/x$, treating the latter as an expansion parameter. Usually, after this reexpansion the stability is restored. In the following, we shall illustrate this possibility by an example, and will suggest a simple trick, giving the answer without the reexpansion, although being equivalent to the latter.

III. SELF-SIMILAR BOOTSTRAP

The procedure of calculating the self-similar approximations (16), starting from a perturbative series $p_k(x)$ is now completely defined. The renormalized quantity $f_k^*(x)$ must be, by construction involving the stability properties, a much better approximation to the sought function $f(x)$ than the initial perturbative series $p_k(x)$. To improve the accuracy, we may repeat the self-similar renormalization, applying it to other series that are left in Eq. (16).

For illustrating this *multiple self-similar renormalization*, consider explicitly a perturbative series

$$p_k(x)=\sum_{n=0}^{k}a_n x^n, \quad a_0\neq 0, \tag{17}$$

containing integer powers of x, although, as is mentioned above, the procedure works for arbitrary noninteger powers. Following Sec. II, we write the algebraic transform

$$P_k(x,s)=\sum_{n=0}^{k}a_n x^{n+s} \tag{18}$$

of Eq. (17). As is seen, transform (18) corresponds to an effectively higher perturbation order, $k+s$, as compared to the initial series (17), of order k. Equation (3) for the expansion function $x(f,s)$ now reads

$$P_0(x,s)=a_0x^s=f, \tag{19}$$

from where

$$x(f,s)=\left(\frac{f}{a_0}\right)^{1/s}. \tag{20}$$

The cascade-trajectory points in Eq. (4) become

$$y_k(f,s)=\sum_{n=0}^{k}a_n\left(\frac{f}{a_0}\right)^{n/s+1}. \tag{21}$$

The velocity field (10) is written

$$v_k(f,s)=a_k\left(\frac{f}{a_0}\right)^{1+k/s}. \tag{22}$$

Calculating the evolution integral (11), with condition (12), we obtain approximation (8) in the form

$$P_k^*(x,s)=P_{k-1}(x,s)\left(1-\frac{ka_k}{sa_0^{1+k/s}}P_{k-1}^{k/s}(x,s)\right)^{-s/k}. \tag{23}$$

The stabilizer $s_k(x)$ is to be found from the minimization of the multiplier

$$m_k(x,s)=\sum_{n=0}^{k}\frac{a_n}{a_0}\left(1+\frac{n}{s}\right)x^n \tag{24}$$

from Eq. (14). Then we obtain the self-similar approximation (16)

$$f_k^*(x)=p_{k-1}(x)\left(1-\frac{ka_k}{sa_0^{1+k/s}}x^k p_{k-1}^{k/s}(x)\right)^{-s/k}, \tag{25}$$

where $s=s_k(x)$.

In this way, the self-similar renormalization led us from the initial perturbative series (17) to the renormalized approximation (25). The latter contains a perturbative series $p_{k-1}(x)$ of lower order than the initial $p_k(x)$. This can be written as the relation

$$f_k^*(x)=F_k(x,p_{k-1}(x)) \tag{26}$$

showing that Eq. (25) depends on x and $p_{k-1}(x)$. We may repeat the procedure renormalizing $p_{k-1}(x)$ and getting $f_{k-1}^*(x)$. With such a double renormalization, we come from Eq. (25) to

$$f_k^{**}(x)=F_k(x,f_{k-1}^*(x))=F_k(x,F_{k-1}(x,p_{k-2}(x))). \tag{27}$$

Here the doubly renormalized $f_k^{**}(x)$ is expressed through $p_{k-2}(x)$. Repeating the self-similar renormalization k times, we obtain the k-fold self-similar approximation

$$f_k^{*\cdots*}(x)=F_k(x,F_{k-1}(x,\ldots(F_1(x,a_0))\ldots)), \tag{28}$$

where we took into account that $p_0(x)=a_0$. It may happen that Eq. (28) contains other power series. Then we may renormalize them as well. When all power series are renormalized, so that none of them is left unrenormalized, we have

$$f_k^{*\cdots*}(x)\to\tilde{f}_k(x). \tag{29}$$

This complete procedure of the self-similar renormalization of all power series, resulting in an expression $\tilde{f}_k(x)$ containing none of them, will be called the *self-similar bootstrap.*

IV. MULTIPLE EXPONENTIALS

There are particular cases of the multiple self-similar renormalization yielding a nice exponential representation of the resulting formulas. Here we consider one such a sufficient condition when all coefficients at perturbative powers are positive, $a_n>0$. Then the minimum of the multiplier (24) is realized at $s\to\infty$. Taking this limit in Eq. (25) gives

$$f_k^*(x)=p_{k-1}(x)\exp\left(\frac{a_k}{a_0}x^k\right). \tag{30}$$

Repeating the renormalization in line with Eqs. (26) and (27), we obtain

$$f_k^{**}(x)=p_{k-2}(x)\exp\left(\frac{1}{a_0}(a_{k-1}x^{k-1}+a_kx^k)\right). \tag{31}$$

Continuing, we obtain the k-fold approximation (28) in the form

$$f_k^{*\cdots*}(x)=a_0\exp\left(\frac{1}{a_0}(a_1x+a_2x^2+\cdots+a_kx^k)\right). \tag{32}$$

As we see, the k-times repeated self-similar renormalization does not deliver us from the power series. Really, the k-star approximation (32) is expressed through a part of the initial perturbation series (17), namely, through

$$p_k(x)-a_0=\sum_{n=1}^{k}a_nx^n.$$

With the notation

$$p_k'(x)\equiv\sum_{n=0}^{k}a_n'x^n, \tag{33}$$

in which $a_n'\equiv a_{n+1}$, $n=0,1;2,\ldots,k$, we may rewrite Eq. (32) as

$$f_k^{*\cdots*}(x)=a_0\exp\left(\frac{x}{a_0}p_{k-1}'(x)\right). \tag{34}$$

The power series $p_{k-1}'(x)$ can be renormalized in our standard way, giving the corresponding self-similar approximation

$$f_{k-1}'(x)=a_0'\exp\left(\frac{x}{a_0'}p_{k-2}''(x)\right), \tag{35}$$

in which

$$p_k''(x)\equiv\sum_{n=0}^{k}a_n''x^n,\quad a_n''\equiv a_{n+2}. \tag{36}$$

With this renormalization in mind, we transform Eq. (34) into

$$f_k^{*\cdots*}(x)=a_0\exp\left(\frac{x}{a_0}f_{k-1}'(x)\right). \tag{37}$$

Combining (35) and (37), we have

$$f_k^{*\cdots*}(x)=a_0\exp\left[\frac{x}{a_0}a_1\exp\left(\frac{x}{a_1}p_{k-2}''(x)\right)\right]. \tag{38}$$

Converting k times all power series in the exponentials, with the use of the notation

$$b_0=a_0,\quad b_k=\frac{a_k}{a_{k-1}},\quad k=1,2,\ldots, \tag{39}$$

we obtain the *bootstrap self-similar approximation*

$$\tilde{f}_k(x)=b_0\exp(b_1x\exp\{b_2x\exp[\ldots b_{k-1}x\exp(b_kx)]\}\ldots), \tag{40}$$

as is discussed in Eq. (29).

In the case of small $x \to 0$, expression (40) yields

$$\tilde{f}_k(x) \simeq c_0 + c_1 x + c_2 x^2 + c_3 x^3, \tag{41}$$

with the coefficients

$$c_0 = b_0, \quad c_1 = b_0 b_1, \quad c_2 = b_0 b_1 (b_2 + \tfrac{1}{2} b_1),$$

$$c_3 = b_0 b_1 (b_2 b_3 + \tfrac{1}{2} b_2^2 + b_1 b_2 + \tfrac{1}{6} b_1^2).$$

Substituting Eq. (39) here, we have

$$c_0 = a_0, \quad c_1 = a_1, \quad c_2 = a_2 + \frac{a_1^2}{2a_0},$$

$$c_3 = a_3 + \frac{a_2^2}{2a_1} + \frac{a_1 a_2}{a_0} + \frac{a_1^3}{6a_0^2}.$$

This shows that the asymptotic behavior of Eqs. (41) and (17), as $x \to 0$, coincides up to the linear terms. For the higher-order terms, $a_n \neq b_n$ for $n \geq 2$. Such a renormalization of the higher-order expansion coefficients is typical of the self-similar approximation theory [2–4]. This renormalization allows us to extend the region of applicability of self-similar approximations, with respect to a variable x, as compared to the initial perturbative series.

It is worth mentioning that the multiple, or continued, exponentials of type (40) have been studied in mathematical literature for more than two centuries, since Euler; a number of references can be found in Refs. [36,37]. We derived form (40) following the multiple self-similar renormalization for the power series (17) with positive coefficients. Of course, for other sets of coefficients in Eq. (17) the final bootstrap approximation will not necessarily take the form of a multiple exponential, as in Eq. (40), but will be a mixture of exponentials and fractions, each expression being conditioned by the principle of maximal stability (15).

Another way of obtaining a multiple exponential could be as follows. Consider a sequence $\{\varphi_k(x)\}$ of the terms

$$\varphi_1(x) = b_0 \exp(b_1 x),$$

$$\varphi_2(x) = b_0 \exp[b_1 x \exp(b_2 x)],$$

and so on, where the coefficients b_k are given by Eq. (39). At the k step of the sequence $\{\varphi_k(x)\}$ we come to Eq. (40). However, this way is justified if the approximation cascade corresponding to the sequence $\{\varphi_k(x)\}$ possesses a stable trajectory leading to a quasifixed point representing Eq. (40). This approximation cascade can be constructed in the standard way [32,34]. From the equation $\varphi_1(x) = f$ we find $x = x(f)$, which is

$$x(f) = \frac{a_0}{a_1} \ln\left(\frac{f}{a_0}\right).$$

Then we define the approximation cascade as is described in Sec. II. The corresponding trajectory $\{z_k(f)\}$, consisting of the terms

$$z_k(f) \equiv \varphi_k(x(f)),$$

is bijective to the approximation sequence $\{\varphi_k(x)\}$.

The stability of the trajectory $\{z_k(f)\}$ is checked by calculating the multipliers

$$M_k(x) \equiv \left(\frac{\partial z_k(f)}{\partial f}\right)_{f = \varphi_1(x)}, \tag{42}$$

and analyzing them with respect to the stability condition $|M_k(x)| < 1$.

At the end of this section let us remark that here we have used the word ''bootstrap'' in the generally accepted meaning, as a kind of a completely self-consistent procedure permitting us to construct an explicit solution to a complicated problem. This term in the close meaning was used, for example, in constructing a self-consistent distribution over particle masses in high-energy physics [38], and also in constructing the S matrices for two-dimensional conformal field theories [39].

V. ZERO-DIMENSIONAL ANALOGS OF FIELD THEORIES

A. Nondegenerate vacuum

Consider the partition function of a zero-dimensional anharmonic model (see, e.g., [32]) represented by the integral

$$J(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-x^2 - g x^4)\, dx, \tag{43}$$

with the integrand possessing a single ''vacuum'' state, located at the point $x = 0$. The expansion of this integral in powers of the coupling parameter g, around the vacuum state leads to divergent series,

$$J(g) \sim a + bg + cg^2 + dg^3 + hg^4 + \cdots, \tag{44}$$

$$a = 1, \quad b = -\tfrac{3}{4}, \quad c = \tfrac{105}{32}, \quad d = -\tfrac{3465}{128}, \quad h = \tfrac{675\,675}{2048}.$$

Here we apply the self-similar bootstrap renormalization, guided by a desire to perform as many renormalization steps as possible. We write down the following set of approximations to the quantity $J(g)$, analogous to the general form (17):

$$J_0(g) = a,$$

$$J_1(g) = a + bg,$$

$$J_2(g) = a + bg + cg^2, \tag{45}$$

$$J_3(g) = a + bg + cg^2 + dg^3,$$

$$J_4(g) = a + bg + cg^2 + dg^3 + hg^4,$$

together with the following local multipliers, which can be found from the general representation (24):

$$m_1(g,s) = 1 + \frac{b}{a} \frac{1+s}{s} g, \tag{46}$$

$$m_2(g,s)=m_1(g,s)+\frac{c}{a}\frac{2+s}{s}g^2,$$

$$m_3(g,s)=m_2(g,s)+\frac{d}{a}\frac{3+s}{s}g^3,$$

$$m_4(g,s)=m_3(g,s)+\frac{h}{a}\frac{4+s}{s}g^4.$$

Analysis of Eqs. (46) shows that, in the last three cases, the most stable trajectories are realized at $s\to\infty$, and that in the first case, $s\to\infty$ also corresponds to a stable trajectory. Therefore the starting four steps of the self-similar bootstrap renormalization can be safely performed in the exponential form, leading to the intermediate renormalized quantity

$$J_4^{****}(g)=a\exp\left(\frac{1}{a}(bg+cg^2+dg^3+hg^4)\right). \quad (47)$$

We write down a set of approximations to the quantity $J'(g)=bg+cg^2+dg^3+hg^4$, appearing in the exponential of this expression,

$$J_1'(g)=bg,$$

$$J_2'(g)=bg+cg^2,$$

$$J_3'(g)=bg+cg^2+dg^3,$$

$$J_4'(g)=bg+cg^2+dg^3+hg^4,$$

with the following local multipliers:

$$m_2'(g,s)=1+\frac{c}{b}\frac{2+s}{1+s}g,$$

$$m_3'(g,s)=m_2'(g,s)+\frac{d}{b}\frac{3+s}{1+s}g^2, \quad (48)$$

$$m_4'(g,s)=m_3'(g,s)+\frac{h}{b}\frac{4+s}{1+s}g^3.$$

Analysis of Eq. (48) leads us to the conclusion that the exponential renormalization is optimal at every step and, following the standard prescriptions of Sec. IV, we transform Eq. (47) into

$$J_4^{****}(g)=a\exp\left[\frac{b}{a}g\exp\left(\frac{1}{b}(cg+dg^2+hg^3)\right)\right]. \quad (49)$$

Our routine procedure now requires us to renormalize the quantity $J''(g)=cg+dg^2+hg^3$, using the approximations

$$J_1''(g)=cg,$$

$$J_2''(g)=cg+dg^2,$$

$$J_3''(g)=cg+dg^2+hg^3,$$

and analyzing the following multipliers:

$$m_2''(g,s)=1+\frac{d}{c}\frac{2+s}{1+s}g,$$

$$m_3''(g,s)=m_2''(g,s)+\frac{h}{c}\frac{3+s}{1+s}g^2.$$

We conclude that the most stable trajectory corresponds, in both cases, to the exponential summation, leading to the intermediate formula

$$J_4^{****}(g)=a\exp\left\{\frac{b}{a}g\exp\left[\frac{c}{b}\exp\left(\frac{1}{c}(dg+hg^2)\right)\right]\right\}. \quad (50)$$

The last step of the procedure, applied to the quantity $J'''(g)=dg+hg^2$, with the approximations set, is

$$J_1'''(g)=dg,$$

$$J_2'''(g)=dg+hg^2,$$

and, with the multiplier

$$m_2'''(g,s)=1+\frac{h}{d}\frac{2+s}{1+s}g,$$

should again be performed with $s\to\infty$; the bootstrap program is completed:

$$\tilde{J}_4(g)=a\exp\left[\frac{b}{a}g\exp\left(\frac{c}{b}\exp\left\{\frac{d}{c}g\left[\exp\left(\frac{h}{d}g\right)\right]\right\}\right)\right]. \quad (51)$$

Similar expressions follow when fewer terms from the initial expansion are taken into account:

$$\tilde{J}_3(g)=a\exp\left\{\frac{b}{a}g\exp\left[\frac{c}{b}\exp\left(\frac{d}{c}g\right)\right]\right\},$$

$$\tilde{J}_2(g)=a\exp\left[\frac{b}{a}g\exp\left(\frac{c}{b}g\right)\right],$$

$$\tilde{J}_1(g)=a\exp\left(\frac{b}{a}g\right).$$

We already pointed out that the last expression corresponds to a stable, but not to an optimal, i.e., the most stable, trajectory. Analyzing $m_1(g,s)$, we obtain the optimally renormalized expression

$$\tilde{J}_{1o}(g)=a\left(1-\frac{b}{as(g)}g\right)^{-s(g)}, \quad s(g)=\frac{-bg}{a+bg}. \quad (52)$$

At point $g=1$, the following numbers are generated by the sequence $\tilde{J}_i, i=2,3,\ldots$:

$$\tilde{J}_1(1)=0.472(\tilde{J}_{1o}=0.512), \quad \tilde{J}_2(1)=0.991,$$

$$\tilde{J}_3(1)=0.473, \quad \tilde{J}_4(1)=0.991.$$

We observe two subsequences, with odd and even numbers, with values practically unchanged within each subsequence, probably embracing the correct result from below and above,

respectively. We can suspect that the corresponding sequence $\widetilde{J}_i$ possesses the two competing unstable quasifixed points, i.e., behaves chaotically, and, in such a situation, it is appropriate to use a self-similar approximation smoothed by the Cesaro averaging procedure [32]; i.e., in our case, simply to take the average over the two neighboring members of each subsequence.

This conjecture is supported by an analysis of the corresponding sequence of the multipliers for the sequence $\widetilde{J}_i$, as discussed in Sec. IV. From the initial approximation

$$\widetilde{J}_1(g)=f,$$

one can find the expansion function

$$g=\frac{a}{b}\ln\left(\frac{f}{\sqrt{\pi}a}\right),$$

and, after the routine transformations, the following expressions for the multipliers (42) can be obtained:

$$M_1(g)\equiv 1,$$

$$M_2(g)=\Phi_2(g)\Psi_2(g),$$

$$M_3(g)=\Phi_3(g)\Psi_3(g),$$

$$M_4(g)=\Phi_4(g)\Psi_4(g),$$

where

$$\Phi_2(g)=\frac{\widetilde{J}_2(g)}{a}\exp\left(\frac{c}{b}g\right),$$

$$\Phi_3(g)=\frac{\widetilde{J}_3(g)}{a}\exp\left[\frac{c}{b}g\exp\left(\frac{d}{c}g\right)\right],$$

$$\Phi_4(g)=\frac{\widetilde{J}_4(g)}{a}\exp\left\{\frac{c}{b}g\exp\left[\frac{d}{c}g\exp\left(\frac{h}{d}g\right)\right]\right\}$$

and

$$\Psi_2(g)=b+cg,$$

$$\Psi_3(g)=b+cg\exp\left(\frac{d}{c}g\right)+dg^2\exp\left(\frac{d}{c}g\right),$$

$$\Psi_4(g)=b+cg\exp\left[\frac{d}{c}g\exp\left(\frac{h}{d}g\right)\right]+dg^2\exp\left(\frac{h}{d}g\right)\exp\left[\frac{d}{c}g\exp\left(\frac{h}{d}g\right)\right]+hg^3\exp\left(\frac{h}{d}g\right)\exp\left[\frac{d}{c}g\exp\left(\frac{h}{d}g\right)\right].$$

The following values are obtained at $g=1$:

$$M_1=1,\quad M_2=-0.089,\quad M_3=1.008,\quad M_4=-0.089,$$

supporting our initial guess that the approximation cascade behaves chaotically. After the Cesaro averaging, the sought value at $g=1$ equals, say, $[\widetilde{J}_3(1)+\widetilde{J}_4(1)]2=0.731$, deviating from the exact value 0.772 with the percentage error -5.228%, an acceptable accuracy if we remember that the initial expansion (44) gives the percentage error $\sim 10^4\%$. With the optimized $\widetilde{J}_{1o}$, we obtain an even better estimate 0.752 for the sought value, with the percentage error equal to -2.668%.

B. Double-degenerate vacuum

Consider the integral

$$I(g)=\int_{-\infty}^{\infty}\exp(x^2-gx^4)dx, \tag{53}$$

representing zero-dimensional field theory, with the integrand possessing the two maxima, located at the points

$$\overline{x}=\pm\frac{1}{\sqrt{2g}},$$

where g plays the role of coupling constant. We intend to estimate this integral in the region of intermediate couplings $g\sim 1$. It was pointed out in [40] that any conventional expansion, in powers of g or g^{-1}, is not sufficient, since it does not take into account the existence of those degenerate maxima, corresponding to the double-degenerate ''vacuum.''

Within the framework of D-dimensional field theories, the existence of a degenerate vacuum is taken into account, e.g., by means of the zero-energy instanton–anti-instanton solutions, and all further corrections to observables come from the excitations above the instanton–anti-instanton background, and from interaction of all those quasiparticles. In our case we take into account the double-degenerate vacuum by means of the shift

$$x=\overline{x}-X,$$

then expand the integral around the two saddle points and apply the self-similar renormalization to the resulting asymptotic expansion in powers of a small parameter $g^{1/2}$, continuing it to the region of $g\sim 1$. Following these prescriptions, represent the integrand in the vicinity of one of the saddle points in the form

$$\exp(x^2-gx^4)\approx\exp\left(\frac{1}{4g}\right)\exp(-2X^2)\exp[A(g)X^3]$$

$$\approx\exp\left(\frac{1}{4g}\right)\exp(-2X^2)[1+A(g)X^3+\cdots],$$

and perform the integration, so that

$$I(g)\approx 2\exp\left(\frac{1}{4g}\right)(a+bg^{1/2}+\cdots),$$

$$a=\sqrt{\pi}2^{-3/2},\quad b=2^{-3/2}. \tag{54}$$

Applying a self-similar renormalization, we readily obtain

$$I^*(g)=2a\exp\left(\frac{1}{4g}\right)\exp\left(\frac{bg^{1/2}}{a}\right). \tag{55}$$

Despite the absence of dynamics in the zero-dimensional case, $I^*(g)$ consists of two factors of different nature: one of them is nonanalytic in the coupling constant, resembling the well-known ''instanton'' term within the framework of nontrivial D-dimensional field theories; the second is analytic in $g^{1/2}$, and resembles a contribution from the excitations above the instanton–anti-instanton background.

The percentage error for the renormalized $I^*(1)$, calculated with respect to the exact $I(1)=2.762$, is 2.462%, and considerable improvement is achieved compared to the percentage error of the perturbative expansion (54), equal to -8.834%.

VI. STRONG-COUPLING REGIME

A. Zero-dimensional case

Let us apply to Eq. (43) the so-called ''strong-coupling'' expansion, in powers of $1/g$, with a quartic term of the integrand taken as a zero approximation, representing the integrand of Eq. (43) as follows:

$$\exp(-x^2-gx^4)\approx\exp(-gx^4)\left(1-x^2+\frac{x^4}{2}+\cdots\right).$$

After integration, we obtain the expansion in inverse powers of the coupling constant

$$J(g)\approx ag^{-1/4}+bg^{-3/4}+cg^{-5/4}+\cdots,$$

$$a=\frac{1.813}{\sqrt{\pi}},\quad b=\frac{-0.612}{\sqrt{\pi}},\quad c=\frac{0.227}{\sqrt{\pi}}. \tag{56}$$

We write the following consecutive approximations to the quantity $J(z)$, where $z=g^{-1/4}$:

$$J_1(z)=az,$$

$$J_2(z)=az+bz^3,$$

$$J_3(z)=az+bz^3+cz^5;$$

and the multiplier $m_2(z,s)=1+(b/a)[(3+s)/(1+s)]z^2$ reaches its minimal zero value at $s=0.019$, being much smaller than the minimal value of the corresponding multiplier $m_3(z,s)=m_2(z,s)+(c/a)[(5+s)/(1+s)]z^4$. Therefore, the renormalized quantity $J_2^*(z)$, will correspond to a more stable trajectory than $J_3^*(z)$. Following the standard prescriptions of Sec. III, we obtain

$$J_2^*(x)=ax\left(1-\frac{2b}{a[1+s(x)]}x^2\right)^{-[s(x)+1]/2},$$

$$s(x)=-\frac{a+3bx^2}{a+bx^2}.$$

The percentage error for the renormalized quantity $J_2^*(1)$, calculated with respect to the exact $J(1)=0.772$, is 2.266%, and a considerable improvement is reached compared to the percentage error of the perturbative expansion (53) with only starting two terms taken into account, equal to -12.208%.

We represent Eq. (43) in a slightly different form,

$$J(g)=g^{-1/4}(a+bg^{-1/2}+cg^{-1}+\cdots)\equiv g^{-1/4}[a+\overline{J(g)}], \tag{57}$$

and write the following set of approximations to $\overline{J(g)}$, using the variable $y=g^{-1/2}$:

$$\overline{J_1}(y)=by,$$

$$\overline{J_2}(y)=by+cy^2.$$

The multiplier $m_2(y,s)=1+(c/b)[(2+s)/(1+s)]y$ is minimal at $s=0$, and $|m_2(y,0)|<1$. The renormalized quantity $\overline{J_2}^*(y)$ can be readily written down,

$$\overline{J_2}^*=\frac{by}{1-\frac{c}{b}y}.$$

Now recalculating $J^*(g)$, we obtain $J^*(1)=0.771$, with the percentage error -0.13%, much better than the percentage error 4.386%, given by the initial expansion (57). Even at small $g=0.21$, the percentage error given by the renormalized expression remains less than 1%; at the same time, the percentage error given by the initial expression reaches 43.538%.

B. One-dimensional case

Consider the dimensionless ground-state energy $e(g)$ of the celebrated quantum one-dimensional quartic anharmonic oscillator, closely connected to the so-called φ^4 model in the quantum field theory (see, e.g., [3]). Here g stands for the dimensionless coupling constant, expressed through the parameters entering the Hamiltonian of the system by the known relation (see, e.g., [3,41]). The asymptotic expansion for $e(g)$ in the strong-coupling limit, corresponding to $g\to\infty$, is known (see, e.g., [41]) in the following form:

$$e(g)\cong ag^{1/3}+bg^{-1/3}+cg^{-1},$$

$$a=0.667\,986,\quad b=0.143\,67,\quad c=-0.0088. \tag{58}$$

Let us, using the experience gained while considering the strong-coupling limit of the zero-dimensional field theory, renormalize the last two terms of the expansion (58). Using the notation $y=g^{-1/3}$, we write the following set of approximations for the quantity $\overline{e}=e-ag^{1/3}$:

$$\overline{e_1}(y)=by,$$

$$\overline{e_2}(y)=by+cy^3,$$

with the multiplier $m_2(y,s)=1+(c/b)[(3+s)/(1+s)]y^2$, possessing minimal value at $s=0$, when $g\geqslant 0.1$. The renormalized expression can be obtained following the standard prescriptions of Sec. III. Returning to the initial variable, we obtain

$$e^*(g)=ag^{1/3}+\frac{b^{3/2}}{\sqrt{bg^{2/3}-2c}}. \tag{59}$$

An accuracy given by $e^*(g)$, can be elucidated by comparison with the ''exact'' numerical results (see, e.g., [41]). At $g=0.3$, the percentage error, given by Eq. (59), is equal to -0.099%, at $g=1$ it is -0.022%, and at $g=200$ it is practically zero. To our knowledge, these results are better than those obtained by other analytical methods. On the other hand, at small $g=0.001$, an accuracy of formula (59) is by far inferior, compared to many other analytical methods. The reason can be understood if we notice that $e^*(0)=0.410\,48$, strongly deviating from the known value one-half, but being much better than the infinite value predicted by the initial expansion (58). We conclude by remarking that using the effective time t^* as an optimization parameter, determined from the condition $e^*(0)=\frac{1}{2}$, one can achieve better agreement of the renormalized formulas with the exact results in the region of small coupling constants. For the goal being pursued in the present paper, it is enough to limit the discussion to formulas (59), designed for the intermediate- and strong-coupling regimes.

VII. EQUATION OF STATE

A. System of hard spheres

We demonstrate below how the self-similar bootstrap can be applied in the theory of equations of state for simple liquids. Consider the celebrated model system of hard spheres with diameter d [42,43], where the empirical equation of state, connecting pressure p, temperature T, number density n, and reduced density $\rho=\pi n d^2/6$, is known:

$$\frac{p}{nkT}=\frac{1+\rho+\rho^2-\rho^3}{(1-\rho)^3}. \tag{60}$$

The equation of state (60) agrees very well with the molecular dynamics results [42]. On the other hand, the theoretical virial formula according to Percus-Yevick [41,42], is given as follows:

$$\frac{p}{nkT}=\frac{1+\rho+\rho^2-3\rho^3}{(1-\rho)^3}. \tag{61}$$

These two expressions almost coincide at low densities, e.g at $\rho=0.1$, the percentage error of Eq. (61), as compared with Eq. (60), equals -0.18%, while for the intermediate and high densities the agreement becomes very poor; e.g. at $\rho=0.5$, the percentage error is -15.385% and at $\rho=0.8$ it equals -53.112%.

Consider the regular part of Eq. (61), defined as r,

$$r=1+\rho+\rho^2-3\rho^3, \tag{62}$$

as an asymptotic, low-density expansion for the ''true'' regular part $\tilde{r}(\rho)$, and try to continue expression (62) from the region of $\rho\ll 1$, to the region of $\rho\leqslant 1$. It seems reasonable to use only the last three terms from Eq. (62) for renormalization, since the constant term describes the ideal gas behavior, and we are interested in the region of high densities. Let us write the following consecutive approximations to the quantity $\overline{r}=r-1$:

$$\overline{r_1}=\rho,$$

$$\overline{r_2}=\rho+\rho^2,$$

$$\overline{r_3}=\rho+\rho^2-3\rho^3.$$

The multipliers are

$$m_2(\rho,s)=1+\rho\frac{2+s}{1+s},$$

$$m_3(\rho,s)=m_2(\rho,s)-3\rho^2\frac{3+s}{1+s},$$

$$m_1'(\rho,s)=1-3\rho\frac{3+s}{2+s}.$$

It is admissible here to apply the self-similar bootstrap renormalization in the form of the continued exponentials, since at every step of the procedure the exponential summation is performed along the stable trajectory. Following the standard prescriptions of Sec. IV, we obtain

$$\tilde{r}(\rho)=\rho\,\exp[\rho\,\exp(-3\rho)]. \tag{63}$$

The multiplier $M(\rho)$, corresponding to Eq. (63), is given by the expression

$$M(\rho)=\exp[\rho\,\exp(-3\rho)]\exp(-4\rho)(1-3\rho)$$

and is very small at $\rho>\frac{1}{3}$, e.g., $M(0.8)=-0.061$, signaling the robust stability of the sequence of the continued exponentials (63). Recalculating

$$\frac{\tilde{p}}{nkT}=\frac{\tilde{r}(\rho)+1}{(1-\rho)^3} \tag{64}$$

and comparing it to the empirical formula (60), we obtain that, at $\rho=0.1$, the percentage error equals -0.118%; at $\rho=0.5$, the percentage error is -4.061%, and at $\rho=0.8$ it equals -3.516%.

We see that the equation of state (64), obtained from the bootstrap self-similar renormalization, is much better, and more uniformly agrees with the computer experiment, than the initial virial expansion (61), over the entire range of densities. The agreement drastically, by 17 times, improves in the region of high densities.

B. System of hard hexagons

The model of ''hard hexagons'' represents a simple two-dimensional model of impenetrable molecules on the triangular lattice. The model allows an exact solution [44], and the phase transition from the liquid phase existing above the critical value of the so-called activity $z_c=11.0917\ldots$, to the solid phase, existing below z_c, is well studied. The equation of state, describing the dependence of the order parameter R on the activity-related parameter, can be written down in quite a complicated and not very convenient form [44]. On the other hand, the critical value of the density ρ_c at the point of the phase transition is known too, and equals $0.27639\ldots$ [44]. Independently, the high-density expan-

sions of the order parameter in powers of the inverse activity $z' \equiv 1/z$, or in powers of the high-density variable $\rho' = 1 - 3\rho$, were obtained [45]. Their quality is considered very high, since the critical parameters could be determined from them with extremely high accuracy, using the Padé approximants in conjunction with some extrapolation methods [45]. Below, we present simple expressions for the equation of state, obtained as a continuation of the high-density expansions of the order parameter up to the point of phase transition.

The expansion of the order parameter in powers of ρ' is given as follows [45]:

$$R = 1 - 3(\rho')^2 - 9(\rho')^3 - 36(\rho')^4 - 159(\rho')^5. \quad (65)$$

Let us apply to Eq. (65) the bootstrap self-similar renormalization based on the exponential summation at every step, and leading to the equation of state for the system of hard hexagons in the form of the continued exponentials:

$$\widetilde{R}(\rho') = \exp\left[-\rho' \exp\left(3\rho' \exp \times \left\{ 3\rho' \exp\left[4\rho' \exp\left(\frac{53}{12}\rho' \right) \right] \right\} \right) \right]. \quad (66)$$

The function $\widetilde{R}(\rho')$ approaches zero at $\rho_c'^* = 0.170\,005(\pm 1)$, corresponding to $\rho_c^* = 0.276\,665$, and deviating from the exact value by 0.1%. Thus the renormalized equation of state (66) agrees with the initial expansion in the region of $\rho' \ll 1$ by design, and predicts the point of the phase transition with very high accuracy. On the other hand, the form of the continued exponential can be justified *a posteriori*, analyzing the multipliers (42), where it can be shown, after some lengthy, but routine calculations, that $M_5(\rho') \to 0$, in the region of $\rho' \approx \rho_c'^*$; i.e. the stability condition is satisfied along the trajectory, described by the sequence of approximations corresponding to Eq. (66), in the vicinity of the critical point. Similarly, using the known expansion of R up to the fifth-order terms in z', a corresponding equation of state can be obtained. In this case we found the critical $z_c'^* \simeq 12.1803(\pm 1)$, deviating from the exact value by 9.8%.

C. Polymer coil

The expansion factor α of the polymer chain, within the framework of a standard ''beads-on-string'' model, is conveniently represented as a function $\alpha^2 = \alpha^2(z)$ of the parameter $z = 2(3/2\pi)^{3/2} N^{1/2} B/a^3$, where N is the total number of links in the chain, a stands for the typical distance between the beads monomers, and B is the second virial coefficient [46,47]. Below we consider only the case of a polymer coil, corresponding to $z > 0$. In the region of $z \ll 1$, the perturbation theory in powers of z can be developed, and for the short-range potentials one can find [46,47] that

$$\alpha^2 = \alpha^2(z) = 1 + k_1 z + k_2 z^2 + \cdots,$$
$$k_1 \approx 1.28, \quad k_2 = -20.8. \quad (67)$$

One of the important problems in the physics of polymer coils, consists in the continuation of expansion (67) to the region of $z \sim 1$ [46,47]. On the other hand, in the limit of $z \gg 1$, α is related to z by a simple power law,

$$\alpha \sim z^{2\nu - 1}, \quad (68)$$

where the critical index $\nu \geqslant \frac{1}{2}$, can be calculated by different methods [46–48]. We propose below, using self-similar renormalization, a simple way to continue Eq. (67) to the region of arbitrary z, including both known limiting cases and allowing us to estimate ν from the stability condition. The problem of this type was already mentioned above, in Sec. II. From the viewpoint of the applicability of the stability conditions, it is worthwhile to study $\alpha^{-2}(z) \equiv a(z)$, reexpanding it in powers of z, so that

$$a(z) \approx 1 + b_1 z + b_2 z^2 + \cdots, \quad b_1 = -1.28, \quad b_2 = 22.438. \quad (69)$$

The set of approximations to $a(z)$, including the two starting terms from Eq. (69), can be written as

$$a_0 = 1,$$
$$a_1 = 1 + b_1 z,$$

and the expression for the renormalized quantity a_1^* can be readily obtained:

$$a_1^* = \left(\frac{s_1}{s_1 - b_1 z} \right)^{s_1} \Rightarrow \left(\frac{s_1}{-b_1} \right)^{s_1} z^{-s_1} \ (z \to \infty), \quad (70)$$

where the stabilizer s_1 should be positive, if we want to reproduce, in the limit of $z \to \infty$, the correct power-law behavior of $\alpha^2(z)$. A different set of approximations, not including into the renormalization procedure the constant term from Eq. (69), has the form

$$\overline{a_1} = b_1 z,$$
$$\overline{a_2} = b_1 z + b_2 z^2,$$

and applying the standard procedure, we obtain

$$a_2^* = 1 + b_1 z \left(1 - \frac{b_2 z}{b_1 (1 + s_2)} \right)^{-(1+s_2)}$$
$$\Rightarrow \left(\frac{-b_2}{1 + s_2} \right)^{-(1+s_2)} b_1^{2+s_2} z^{-s_2} \quad (z \to \infty). \quad (71)$$

Demanding now that both Eqs. (70) and (71) have the same power-law behavior at $z \to \infty$, we find that

$$s_2 = s_1 = 2(2\nu - 1).$$

Requiring now the fulfillment of the stability criteria for the two approximations (70) and (71) in the form of the minimal-difference condition (see Sec. I), we obtain the condition on the *positive* stabilizer s_1, i.e., s_1 should be determined from the *minimum* of expression A:

$$A=\left|\left[\left(\frac{-b_2}{1+s_1}\right)^{-(1+s_1)} b_1^{(2+s_1)}-\left(\frac{s_1}{-b_1}\right)^{s_1}\right]\right|. \qquad (72)$$

The minimum of Eq. (72) does exist, and is located at the point $s_1\approx 0.5$, Correspondingly, the index ν is equal to 0.625, in reasonable agreement with all other theoretical and experimental estimates of this index [46–48].

As it was pointed out in Sec. II, the results may also be obtained, if the self-similar renormalization is applied to the initial expansion (67), for the sought function $\alpha^2(z)$, although it is formally divergent at $z\rightarrow\infty$, and the stabilizer should become negative to describe this divergence correctly. By simple substitution of the coefficients, and changing the criteria on minimum of Eq. (72) to the *maximum* of the analogous expression K,

$$K=\left|\left[\left(\frac{-k_2}{1+s}\right)^{-(1+s)} k_1^{(2+s)}-\left(\frac{s}{-k_1}\right)^{s}\right]\right|, \qquad (73)$$

with respect to the now *negative* stabilizer $s\equiv 2(1-2\nu)$. One can see that the maximum is located at the point $s=-0.3719$, leading to the very reasonable estimate for the critical index $\nu=0.593$. The final formulas have the following form:

$$\alpha_1^2(z)^*=\left(\frac{s}{s-k_1 z}\right)^s, \qquad (74)$$

$$\alpha_2^2(z)^*=1+k_1 z\left(1-\frac{k_2 z}{k_1(1+s)}\right)^{-(1+s)}. \qquad (75)$$

Both formulas (74) and (75), with $s\approx -0.3719$, can be used as an approximate ''equation of state'' for the polymer in the whole range of the parameter z, satisfying, by design, both known virial and scaling asymptotic expressions.

VIII. CRITICAL TEMPERATURE OF THE 2D ISING MODEL FROM THE EXPANSION AROUND THE DIMENSION ONE

In this section we calculate the critical temperature T_c of the two-dimensional (2D) Ising model starting from the approximate expression obtained within the framework of the so-called quasichemical approximation [42]. This approximation gives T_c as a function of the coordination number z:

$$T_c(z)=\frac{-2}{\ln(1-2/z)}. \qquad (76)$$

Expression (76) correctly describes the limit of the 1D Ising model, with $T_c=0$, and at the infinite dimensionality the limit coincides with the well-known Bragg-Williams result $T_c=z$. The expansion around the latter limit has been widely used, although its accuracy is not too good [49]. We adopt the different approach, expanding Eq. (76), in powers of the parameter $z-2\equiv\Delta$, around its correct, $D=1(z=2)$, limit, and use the self-similar renormalization to continue the expansion valid at $\Delta\ll 1$to the region of $\Delta=2$, corresponding to the 2D Ising model with the quadratic lattice. The expansion of the inverse expression (76), up to the quadratic term in Δ, has the following form:

$$T_c^{-1}(\Delta)\approx\frac{\ln 2}{2}\left[1+\frac{1}{\ln 2}\ln\left(\frac{1}{\Delta}\right)\right]+\tfrac{1}{4}\Delta-\tfrac{1}{16}\Delta^2. \qquad (77)$$

We renormalize separately the logarithmic contribution

$$L(\Delta)=\left[1+\frac{1}{\ln 2}\ln\left(\frac{1}{\Delta}\right)\right], \qquad (78)$$

and power-law contribution

$$P(\Delta)=\tfrac{1}{4}\Delta-\tfrac{1}{16}\Delta^2, \qquad (79)$$

separating, in this way, the effects of long- and short-range contributions to T_c.

The standard prescriptions of Sec. III are fully applicable to expression (78) containing the logarithmic term. Two consecutive approximations to $L(\Delta)$ are

$$L_0(\Delta)=1,$$

$$L_1(\Delta)=1+\frac{1}{\ln 2}\ln\left(\frac{1}{\Delta}\right),$$

with the expansion function $f=\Delta^s$ and the multiplier

$$m_1(\Delta,s)=1-\frac{1}{\ln 2}\left(\ln(\Delta)+\frac{1}{s}\right),$$

equal, at $\Delta=2$, to zero as $s\rightarrow\infty$. The velocity function has the form

$$v(s,f)=-\frac{f}{\ln 2}\frac{\ln f}{s}.$$

Calculating the evolution integral and taking the limit of $s\rightarrow\infty$, we obtain

$$L^*(\Delta)=\Delta^{-1/\ln 2}.$$

The expression for $P^*(\Delta)$ can be readily written down in the case of summation along the stable trajectory, corresponding to $s\rightarrow\infty$,

$$P^*(\Delta)=\tfrac{1}{4}\Delta\exp(-\tfrac{1}{4}\Delta).$$

Recalculating T_c^*, we obtain $T_c^*(\Delta=2)=2.321$.The percentage error equals 2.292%, when compared to the exact Onsager result $T_c=2.269$. It should be remembered that the quasichemical approximation (76) works with the percentage error of 27.149%, and that the initial expansion (77) deviates from the exact result by 76.289%. Also, one of the best known approximate theoretical schemes, known as the Kikuchi method [42], gives a percentage error equal to 6.831%.

IX. TEMPORAL ASYMPTOTES OF THE DIFFUSION EQUATION WITH RANDOM STATIONARY NOISE

A. Poisson spectrum

Consider the diffusion of particles in a medium with randomly distributed traps, whose local density $\alpha(\mathbf{r})$ is described by the non-negative Poisson random field [50–52]. The local particle density $n(\mathbf{r},t)$ in the presence of traps is described by the equation

$$-\frac{\partial}{\partial t}n(\mathbf{r},t)=-\nabla^2 n(\mathbf{r},t)+\alpha(\mathbf{r})n(\mathbf{r},t), \qquad (80)$$

where we set the diffusion coefficient equal to 1.

This problem is formally equivalent to the Schrödinger equation with imaginary time and potential $\alpha(\mathbf{r})$ [50–52]. The eigenvalues of the Schrödinger operator corresponding to Eq. (80), E_i, are non-negative. In addition, the density of states $p(E)$, near the finite fluctuational limit of the spectrum, located at $E=0$, and formed due to the rare fluctuations of the potential with α close to zero in large regions of space, is known explicitly:

$$p(E)=f(E)\exp(-E^{-D/2}), \tag{81}$$

where D is the dimensionality of space [53,54]. Hereafter, for the sake of simplicity, we omit the constant terms in the exponential for the density of states. Also, for $D=1$, the pre-exponential factor $f(E)\sim E^{-3/2}$, is known [53] and, because of this, below we consider only the one-dimensional case. The general solution of Eq. (80) can be readily set down [52] in terms of the eigenfunctions and eigenvalues of the corresponding quantum-mechanical problem:

$$n(r,t)=\sum_i c_i\psi_i(r)\exp(-E_i t), \tag{82}$$

and, considering only contributions from the rare fluctuations of the potential, the mean density $\langle n\rangle$ over the entire volume for $t\to\infty$ can be represented in the form of the integral

$$\langle n(t)\rangle = n_0\int_0^\infty p(E)\exp(-Et)dE, \tag{83}$$

where n_0 describes the initial homogeneous particle distribution. This integral can be evaluated by the method of steepest descent and the leading exponential term $\sim\exp(-t^{1/3})$ can be set down [49–51]. Similar estimates were obtained for arbitrary D [50–52].

Below we will obtain the higher-order contributions to the asymptotic expansion of Eq. (83) near the saddle point $E=(1/2t)^{2/3}$, and, using self-similar renormalization, obtain the leading corrections, as $t\to\infty$, to the pre-exponential factor $f(E)\sim t^{-1}$.

Let us represent E in the vicinity of $\overline{E}$ as $E=\overline{E}+\epsilon$, and expand the expression

$$\Phi(t,\epsilon)\equiv\ln[p(E)\exp(-Et)]$$

in powers of ϵ up to the third-order terms, so that

$$\Phi(t,\epsilon)\approx -at^{1/3}-A(t)\epsilon^2+B(t)\epsilon^3,$$

$$a=3\times 2^{-2/3},\quad A(t)=3\times 2^{-4/3}t^{5/3},\quad B(t)=5\times 2^{-5/3}t^{7/3},$$

and expand $\exp\{B(t)\epsilon^3\}$ in powers of ϵ, so that

$$\exp\{\Phi(t,\epsilon)\}\approx\exp\{-at^{1/3}\}\exp\{-A(t)\epsilon^2\}\times[1+B(t)\epsilon^3+\cdots].$$

Now $\langle n(t)\rangle$ can be written down as follows:

$$\langle n(t)\rangle\sim t^{1/6}\exp\{-at^{1/3}\}[1+bt^{-1/6}],\quad b=0.684, \tag{84}$$

i.e., the corrections to the pre-exponential factor are obtained in the form of an expansion in inverse powers of t, valid as $t\to\infty$. Our aim is to continue this expression to the region of $t\sim 1$.

Apply now the self-similar renormalization to the quantity $\overline{n(t)}=1+bt^{-1/6}$, with the two consecutive approximations

$$\overline{n_0}=1,$$

$$\overline{n_1}=1+bt^{-1/6}.$$

Following the standard prescriptions, the renormalized quantity $\overline{n^*}(t)$ can be obtained:

$$\overline{n^*}(t)=\left(t^{1/6}+\frac{b}{6s(t)}\right)^{6s(t)}, \tag{85}$$

where the stabilizer

$$s(t)=\frac{b}{6t^{1/6}+6b}$$

is defined as the zero of the multiplier

$$m_1(s,t)=1+\frac{bt^{-1/6}(s-\frac{1}{6})}{s}.$$

For the intermediate region $1\ll t<\infty$, the simple expression can be written

$$\overline{n^*}(t)\propto t^{b/6t^{1/6}},$$

which gives the correction to the pre-exponential factor in the form of continued noninteger powers. It is worth noting that, already, the starting terms of the asymptotic expression (85), lead to the approximation-cascade trajectory with zero multiplier.

B. Gaussian spectrum

Consider an equation of the same type as Eq. (80), with the only difference being that the Poisson potential α is replaced by the random potential $U(\mathbf{r})$ with the properties of the Gaussian ''white noise,'' i.e., $\overline{U(\mathbf{r})}=0$, $\langle U(\mathbf{r})U(\mathbf{r}')\rangle\propto\delta(\mathbf{r}-\mathbf{r}')$:

$$-\frac{\partial}{\partial t}n(\mathbf{r},t)=-\nabla^2 n(\mathbf{r},t)+U(\mathbf{r})n(\mathbf{r},t). \tag{86}$$

An equation of this type, but with a noise dependent both on space and time, can be easily transformed to a nonlinear Burgers equation, Kardar-Parisi-Zhang (KPZ) equation, and it also describes some other closely related physical problems [55]. A stationary random potential is not meaningless within the framework of, say, the KPZ equation, where one can think about the stationary, random in space, perturbations of a growing interface. It is also considered in biology as a model for population dynamics in the presence of a random distribution of food [55]. An equation of type (86) with a random potential $U(\mathbf{r})$ can be also transformed to the corresponding Schrödinger equation with imaginary time. Therefore, one can use the known properties of the spectrum

of the corresponding quantum-mechanical problem to study the $t\to\infty$ behavior of the diffusion equation with stationary, randomly distributed sources and sinks. The fluctuational limit of the spectrum is situated now at $E\to-\infty$, with the exponentially small density of states in its vicinity,

$$p(E)=f(E)\exp(-|E|^{2-D/2}), \tag{87}$$

which is formed due to the rare fluctuations of the potential with large negative values, separated from each other by distances much larger than their own sizes [53]. Such fluctuations may be again considered separately. Eigenvalues corresponding to the eigenfunctions localized at these fluctuations are now negative, as distinguished from the case considered above, and the mean density evolution as $t\to\infty$ can be estimated from the following integral:

$$\langle n(t)\rangle = n_0\int_0^\infty p(E)\exp(|E|t)dE. \tag{88}$$

We again use the method of steepest descents, and follow literally the same steps as in Sec. IX A. The saddle point does exist for $1\leqslant D<4$ (except at $D=2$ and 4, where the situation becomes trivial) and is given by the expression

$$|\overline{E}|=\left(\frac{2t}{4-D}\right)^{2/(2-D)}.$$

The leading exponential term has the form

$$\langle n(t)\rangle\sim\exp\{a(D)t^{(4-D)/(2-D)}\},$$

$$a(D)=\left(\frac{2-D}{4-D}\right)\left(2-\frac{D}{2}\right)^{2/(D-2)},$$

with radically different behavior for $D=1$ and 3:

$$\langle n(t)\rangle\sim\exp\{|a(1)|t^3\},\quad D=1 \tag{89}$$

$$\langle n(t)\rangle\sim\exp\{-|a(3)|t^{-1}\},\quad D=3, \tag{90}$$

corresponding to an anomalously fast growth, compared to the simple $\exp(t)$, and anomalously slow decay, compared to $\exp(-t)$, respectively. We think that this difference takes its origin from the principally different properties of the corresponding Schrödinger operator, where it is known that, at $D=1$, all states of the particle are localized, while, at $D=3$, generally speaking, both localized and delocalized states are present [53]. These basic theorems, when applied to the case of diffusion, explain why in one dimension, the random distribution of sources and sinks causes an explosive instability of the density fluctuations, while in three dimension, the disorder can cause only longer decay times for the density fluctuations. Of course, the instability can be cured by nonlinear interactions that should be now taken into account.

At $D=1$, where $f(E)\sim E$ [54], applying the procedure already discussed above, we obtain the expansion in the vicinity of the saddle point:

$$\langle n(t)\rangle\propto\exp\{|a(1)|t^3\}t^{5/2}[1+bt^{-3/2}+\cdots],\quad b=\frac{1}{2\sqrt{\pi}},$$

and the renormalized expression can be readily written, using the same definition for $\overline{n(t)}$ as above,

$$\overline{n^*}(t)=\left(t^{3/2}+\frac{3b}{2s(t)}\right)^{2s(t)/3}\sim t^{3bt^{-3/2}/2},\quad s(t)=\frac{3b}{2(b+t^{3/2})}. \tag{91}$$

At $D=3$, a slightly different situation occurs, since

$$\Phi(t,\epsilon)\approx-|a(3)|t^{-1}+A(t)\epsilon^2-B(t)\epsilon^3,$$

$$A(t)=t^3,\quad B(t)=2t^5,$$

and, in order to guarantee the convergence of the integrals, it is reasonable to expand $\exp[\Phi(t,\epsilon)]$ as follows:

$$\exp\{\Phi(t,\epsilon)\}\approx\exp\{-|a(3)|t^{-1}\}\exp\{-B(t)\epsilon^3\}\times[1+A(t)\epsilon^2+\cdots].$$

For the $\langle n(t)\rangle$ we obtain

$$\langle n(t)\rangle\sim\exp\{-|a(3)|t^{-1}\}t^{-5/3}[1+ct^{-4/3}+\cdots],$$

$$c\approx0.296.$$

The renormalized expression for the pre-exponential factor has the form

$$\overline{n^*}(t)=\left(t^{4/3}+\frac{4b}{3s(t)}\right)^{4s(t)/3}\sim t^{4ct^{-4/3}/3},\quad s(t)=\frac{4c}{3(c+t^{4/3})}. \tag{92}$$

We have demonstrated, in this section, that self-similar renormalization can be applied to dynamical problems as well, generating expressions for the pre-exponential factors in the form of continued noninteger powers.

X. CONCLUSION

Here we suggested a variant of the self-similar approximation theory, permitting an easy and accurate summation of divergent series. The method is based on a power-law algebraic transformation leading to an effective increase of the order of perturbative terms. The powers of this transformation play the role of control functions governing the convergence of renormalized series. These control functions are defined by the principle of maximal stability, i.e., from the minimization of mapping multipliers. Such stabilizing control functions may be called stabilizers.

Another important point of the method is the multiple self-similar renormalization converting all series into closed self-similar expressions. This multiple and complete renormalization is called the self-similar bootstrap. The resulting effective sum of a divergent series can be presented through analytical expressions containing exponentials and fractions, rational or irrational. In particular cases, these can be only exponentials or only fractions, depending on the behavior of control functions which dictate the resulting form. Because of the much larger variety of such resulting forms, the method allows us to present the answers in relatively simple analytical expressions which have, at the same time, quite a high accuracy. The use of several types of functions, such as exponentials and various fractions, distinguishes this method

from, e.g., the Padé approximants, which involve solely rational functions.

In order to prove that the suggested method really gives quite simple and accurate expressions for the effective sums of divergent series, we, first of all, considered several toy models cartooning the generating functionals in field theory or partition functions in statistical physics. By these examples we illustrated that the method works well in different situations, for single- and double-well models, and for weak and strong coupling.

To stress the generality of the method, we applied it to several problems of statistical physics of quite different natures: to constructing the equation of state, to calculating the critical temperature, and to finding the time asymptotics for stochastic dynamical processes. We hope that these various and very different applications demonstrate well the validity of the method.

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